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Airfoil leading-edge suction and energy conservation for compressible flow

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When a flat-plate airfoil at zero angle of attack encounters a vertical gust in an otherwise uniform flow, it experiences a force along the chord. This leading-edge suction force is examined for compressible flow with a time-dependent gust. A simple derivation of the thrust force is based on the fact that the leading edge is a singular point so that the flow here is dominated by the leading-edge dipole strength. From the viewpoint of a fluid-fixed observer the fluid does work on the airfoil, and this energy must come from the incident gust. Demonstrating energy conservation is not surprising, but it gives a better understanding of the relationship between the individual energy terms. The derivation shows that the acoustic energy can be calculated using compact assumptions at low frequency, but that it must be calculated non-compactly at high frequency. For a general gust the work done on the airfoil is shown to equal the energy taken from the fluid, the energy transfer occurring at the leading edge. For a sinusoidal gust the energy contained in the incident gust is shown to equal the sum of the energy remaining in the wake, the work done on the airfoil and the acoustic energy radiated away. The relative proportions of the incident energy going to these three energy types depends on the gust frequency, the acoustic radiation becoming more efficient as the frequency increases. For a fixed gust frequency, the thrust force goes to zero at a Mach number of one, and for an incident gust consisting of vorticity on the airfoil axis, the entire energy of the gust is radiated as acoustic energy at this Mach number.

1. Introduction

When a flat-plate airfoil at zero angle of attack encounters a gust, a leading-edge suction force or thrust on the airfoil is produced. This is known as the Katzmayr effect (1922). In a recent paper Ribner (1993) calculated the suction force for incompressible flow on an airfoil encountering a sinusoidal gust in an otherwise uniform flow. It may seem surprising, at first, that, in addition to a life force, an airfoil will experience a suction force under these conditions, but with a little thought as shown below, the principle becomes quite clear. The present paper extends the analysis to compressible flow and casts the thrust result in a form depending only on the leading-edge loading, valid for any type of gust. Also the mechanism producing the thrust is more clearly defined. It will be seen that it is possible to calculate the force in a quasi-steady manner, greatly simplifying the analysis and interpretation. The pressure at each point of the airfoil surface makes a contribution to the lift force on a flat-plate airfoil, but the only region of importance for the suction force is the flow around the leading edge. A flat-plate airfoil at an angle of attack in a steady compressible flow is first considered. The results when properly expressed are then shown to apply to the unsteady case as well.

By D'Alembert's paradox it is known that there can be no drag force on an airfoil at an angle of attack in smooth steady flow with no viscosity. Any force, due to non-zero circulation around a two-dimensional body, acts in a direction normal to the fluid, not normal to the airfoil as one might expect from the pressure forces (see e.g. Batchelor 1967, p. 405). This paradox is resolved by the observation that there is a suction force on the leading edge caused by the infinite velocity of the fluid flow around the edge. The airfoil pressures act normal to the surface, and cannot produce a force along the chord, except at the leading edge. Because the leading edge has infinitesimal thickness, no finite pressure can contribute to the suction force. Only infinite pressures produced by infinite velocities at the leading edge, can contribute to the thrust, and these velocities are produced only by the force dipoles (bound vorticity) that occur at an infinitesimal distance from the leading edge. Dipoles at other locations on the airfoil can affect the strength of the leading-edge dipoles, and so can indirectly affect the suction force, but once the strength of the leading-edge dipoles is known, the suction force is determined, regardless of the distribution of dipoles at other locations on the surface. The suction force on an airfoil in steady flow will be cast in terms of the instantaneous value of the leading-edge loading. An expression for the thrust for an airfoil encountering a sinusoidal gust in compressible flow then follows immediately by substitution of the leading-edge loading for this case into the steady-flow result. This idea is implicit in the derivation of von Kármán & Burgers (1935), and is also mentioned by Robinson & Laurmann (1956) and Lighthill (1975), all for the incompressible flow case.

For a more complete understanding of the problem, it is useful to consider the conservation of energy, as does Ribner (1993) for the case of incompressible flow. Work is done on the airfoil, and verifying energy conservation gives confidence that the calculation procedures for the various energy terms is correct, and that nothing has been neglected. This also gives a better understanding of the efficiency for conversion of energy of vorticity into acoustic energy. Because the expression given for the thrust depends only on the leading-edge loading and not on the details of the incident disturbance, it is first noted that energy conservation can be established by considering only the flow in the vicinity of the leading edge. Having established the equality between the work done on the airfoil by the suction force and the energy subtracted from the fluid, the division of the fluid energy into incident and wake vortex energy and radiated acoustic energy is considered. It is shown that this division depends on the reduced frequency. Thus, it is not possible to find a division between acoustic energy and wake energy that depends only on the instantaneous value of the leading-edge loading. For a general incident disturbance the energy taken from the fluid equals the work done on the airfoil, but in order to determine the energy remaining in the vorticity a specific incident disturbance must be considered. For small reduced frequency, k, the energy in acoustic radiation is small compared to the energy remaining in the vortex wake, but for large k the reverse is true. Guo (1989) made a similar balance between the energy in the fluid, the acoustic energy and the thrust work for the case of an airfoil cutting a cylindrical jet.

In the derivations below care should be taken to note the coordinate system being used. For calculating the airfoil force and radiated acoustic field, a standard airfoil-fixed coordinate system is used. For the energy calculations, a coordinate system fixed to the ambient fluid is used. Different x symbols for these two coordinate systems could have been used, but the calculations are sufficiently independent that it seemed simpler to use just one symbol, while clearly pointing out the coordinate system used.

Finally, it should be noted that the analysis presented here assumes linearized flow.

The assumptions of linearized flow are violated near the leading edge, and it might appear that analyses such as that presented here would become invalid. This point was addressed to some extent by Jones (1950), Lighthill (1951) and Batchelor (1967, p. 468). The utility of the flat-plate assumption is the simplicity that it gives to the results without introducing anomalous behaviour due to the singularity at the leading edge. Calculation of the flow around an airfoil with a finite leading-edge thickness would be significantly more complex. Since the flat-plate assumption leads to a reasonable approximation to the lift, and since the overall force on a real airfoil in smooth, steady inviscid flow must be normal to the mean flow, the thrust force should be approximately that for a flat plate. However, one would expect a flat-plate analysis, for either the lift or the thrust, to break down for gust disturbances that are smaller than the airfoil leading-edge thickness. Stated another way, so long as the gust disturbance is large compared to the airfoil leading-edge dimension, one would not expect any dramatic difference between results for a finite-thickness airfoil and for a flat plate.

2. Calculation of the thrust

2.1. *Steady flow*

Figure 1 shows an airfoil in a steady flow in a compressible fluid. The force due to the pressure acts normal to the surface, but the overall force is normal to the mean flow vector, neglecting viscous forces. The difference between these two vectors is the force parallel to the chord produced on the airfoil leading edge. For incompressible flow the lift coefficient (Lift/($\frac{1}{2}\rho_0 U^2$ Area)) is $2\pi \sin\alpha$. Compressibility adds a factor of β^{-1} which can be obtained from a Prandtl–Glauert transformation. For a small angle of attack $\sin\alpha$ can be replaced by α. The airfoil chord is c (half-chord b) so that the area is also c for a unit span of airfoil in the spanwise y-dimension. The force on the airfoil is thus

$$L = \rho_0 U^2 c\pi\alpha/\beta, \tag{1}$$

where $\beta^2 \equiv 1-M^2$. The thrust is then

$$T = L\sin\alpha = \pi\rho_0 U^2 c\alpha^2/\beta. \tag{2}$$

By applying a Prandtl–Glauert transform to equations (5.21) and (5.22) of Garrick (1957) or from equations (13) and (16) of Amiet (1974) the airfoil loading (pressure jump) on the airfoil for steady compressible flow is

$$\Delta C_p(x) \equiv \frac{\Delta p}{\frac{1}{2}\rho_0 U^2} = \frac{4\alpha}{\beta}\left(\frac{b-x}{b+x}\right)^{1/2}. \tag{3}$$

Thus, near the leading edge

$$\Delta C_p(x) = 4\alpha\beta^{-1}(c/\epsilon)^{1/2}, \tag{4}$$

where $\epsilon \equiv b+x$ is the distance from the leading edge. Without further consideration it is not clear how one would calculate the thrust on a more general airfoil, say a cambered airfoil, given equation (2) for a flat-plate airfoil. Thus, for a more general result α will be replaced by the leading-edge dipole strength using equation (4); the dipole strength in the vicinity of the leading edge determines the leading-edge flow, and thus the leading-edge thrust for any zero-thickness airfoil. First, define

$$G \equiv \lim_{\epsilon\to 0}[\Delta C_p(x)(\epsilon/c)^{1/2}] = 4\alpha/\beta. \tag{5}$$

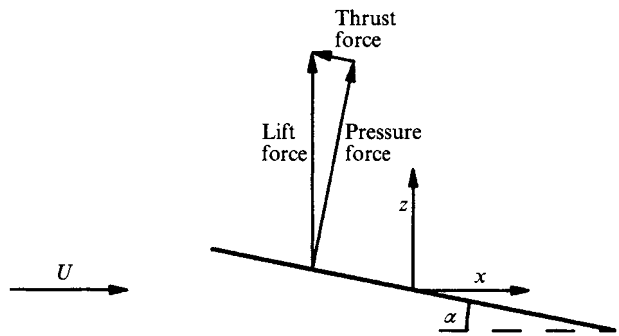

FIGURE 1. Lift on an airfoil in steady flow.

Then, one can define a thrust coefficient

$$C_T \equiv T/(\rho_0 U^2 b) = (\pi\beta/8)\, G^2. \tag{6}$$

This extends Garrick's (1936) result to compressible flow, but with G defined slightly differently.

2.2. *Unsteady flow*

Equation (6) determines the thrust force for the steady case. The interesting point is that this relation also gives the force for the general unsteady case, either due to airfoil motion or to a gust incident on the flat plate. This is because the thrust is totally dependent on the pressure forces at the leading edge, a point of singularity in the flow. Von Kármán & Burgers (1935, pp. 52, 306) demonstrate this principle when they use the suction result derived for steady incompressible flow to show energy conservation for the case of a plunging airfoil, but like Garrick, they express G in terms of the leading-edge vorticity rather than leading-edge loading. To see that the steady result is applicable to the unsteady case, consider the Bernoulli equation in airfoil-fixed coordinates:

$$\partial\phi/\partial t + \tfrac{1}{2}v^2 + p/\rho = \text{constant}. \tag{7}$$

For a small enough region in the vicinity of the leading edge any variations of the constant on the right-hand side can be ignored since the scale of the incident disturbance is finite while the leading-edge thickness is infinitesimal. Also, whereas the perturbation velocity u is large, $O(\epsilon^{-1/2})$ at a distance ϵ from the leading edge, the potential $\phi(\epsilon)$ is $O(\epsilon^{1/2})$ since $u = \partial\phi/\partial x$. Thus, $\partial\phi/\partial t$ can be ignored and the steady Bernoulli equation applies. That is, the pressure can be calculated from the instantaneous velocity field.

An example of this principle is that the perturbation pressure and velocity near the leading edge are related by the steady relation $p = \rho_0 U u$. Noting that $\Delta p = 2p$, equation (29) of Amiet (1990) gives for the airfoil pressure due to a sinusoidal gust in incompressible flow

$$p_u(x) = -\rho_0 U w_0 \left(\frac{b-x}{b+x}\right)^{1/2} S(k)\, e^{i\omega t}. \tag{8}$$

The subscript u denotes the upper surface, although the only difference between upper and lower surfaces is the sign of p. $S(k)$ is the Sears function, defined as

$$S(k) = \frac{2}{\pi k}[H_0^{(2)}(k) - iH_1^{(2)}(k)]^{-1}, \tag{9}$$

where $H_n^{(2)}(k)$ is the Hankel function of the second kind. Expanding equation (33) of Amiet (1990) for small x and comparing with equation (8) gives for the vorticity near the leading edge

$$\tfrac{1}{2}\rho_0 U\gamma(x) = -\rho_0 U u_u(x) = p_u(x), \tag{10}$$

which is the relation between p and u for steady flow. Another illustration of this leading-edge behaviour can be seen from the general relation

$$\phi(x,z) = -\frac{1}{\rho_0 U}\int_{-\infty}^{x} p\left(\xi, z, t-\frac{x-\xi}{U}\right)\mathrm{d}\xi, \tag{11}$$

which is a form of the momentum equation. Taking the x-derivative gives two terms, but the dominant term near the leading edge is $u = -p(x,z,t)/(\rho_0 U)$. Equation (6) uses the surface loading rather than the strength of the bound vorticity at the leading edge which Garrick (1936) uses. At the leading edge Δp and γ are related by $\Delta p = \rho_0 U\gamma$, but expressions for Δp are more readily available than those for γ.

Thus, although equation (6) was derived for steady flow, it is a general result for determining the airfoil thrust if the airfoil loading near the leading edge is known. For the specific case of an airfoil in compressible flow encountering a sinusoidal gust, $w_0 \exp(\mathrm{i}\omega t - \mathrm{i}kx)$, equations (13) and (16) of Amiet (1974) give for the leading-edge loading

$$\Delta p(x) = -2\rho_0 U w_0 \beta^{-1}(c/\epsilon)^{1/2} S(k^*)\, \mathrm{e}^{\mathrm{i}\omega t + \mathrm{i}k^* f(M)}, \tag{12}$$

where $k \equiv \omega b/U$, $k^* \equiv k/\beta^2$, $\epsilon \equiv b+x$ and $f(M)$ is a function of Mach number given by Amiet (1974) that will drop out of this analysis. Thus, for a sinusoidal gust

$$G_s = 4w_0(\beta U)^{-1} S(k^*)\, \mathrm{e}^{\mathrm{i}\omega t + \mathrm{i}k^* f(M)} \tag{13}$$

and equation (6) gives for the average thrust

$$\frac{\bar{T}_s}{\rho_0 w_0^2 b} = \frac{\pi}{\beta}|S(k^*)|^2, \tag{14}$$

agreeing with the result of Ribner (1993) when $M = 0$. A factor of $\frac{1}{2}$ comes from taking the time average of $\cos^2(\omega t)$ contained in G^2. Equation (6) is really the more general equation, however, since it remains valid regardless of the time variation in the problem whereas equation (14) is based on an explicit leading-edge loading.

3. Energy subtracted from the flow

3.1. *Expressions for the energy*

As noted by Ribner, the thrust force on the airfoil subtracts energy from the flow; that is, the fluid does work on the airfoil when considered from a coordinate system fixed to the ambient fluid. For the incompressible flow case, the shed vorticity tends to cancel the incident vorticity. It is of interest to make similar calculations for the compressible case, although now there is potential energy in addition to the kinetic energy. Whereas an airfoil-fixed coordinate system was used to calculate the airfoil thrust, a fluid-fixed coordinate system is used here to calculate the kinetic and potential energies of the fluid. These are given by Pierce (1981) as

$$E_K = \tfrac{1}{2}\rho_0 v^2, \quad E_P = \tfrac{1}{2}p^2/(\rho_0 c_0^2). \tag{15}$$

The fluid velocities and pressure are related to the velocity potential by

$$v = \nabla\phi, \quad p = -\rho_0 \partial\phi/\partial t. \tag{16}$$

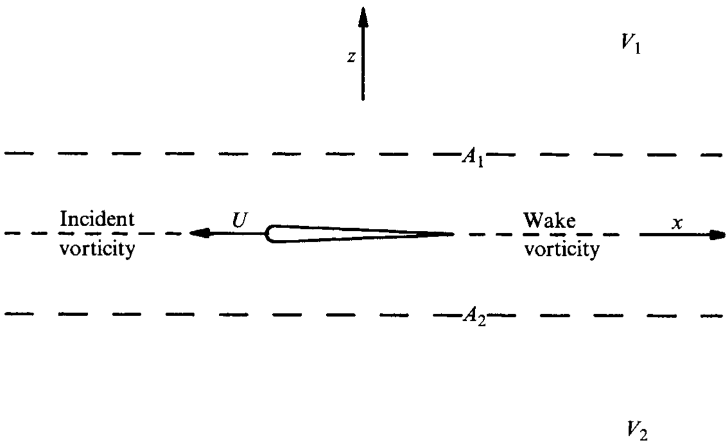

FIGURE 2. Control areas A_1 and A_2 for calculating the energy flow.

The kinetic and potential energies are added and integrated over the upper and lower fluid volumes, V_1 and V_2, together constituting the entire (x, z)-plane with the exclusion of the region around the x-axis as shown in figure 2. This volume is excluded in order to exclude the airfoil sound source, but the volume excluded is made vanishingly small. The total energy E is then

$$\frac{2}{\rho_0}E = \int_V \left[\nabla\phi\cdot\nabla\phi+\frac{1}{c_0^2}\left(\frac{\partial\phi}{\partial t}\right)^2\right]\mathrm{d}x\,\mathrm{d}z = \int_V\left[\nabla\cdot(\phi\nabla\phi)-\phi\nabla^2\phi+\frac{1}{c_0^2}\left(\frac{\partial\phi}{\partial t}\right)^2\right]\mathrm{d}x\,\mathrm{d}z. \quad (17)$$

Introducing the wave equation

$$c_0^2\nabla^2\phi-\partial^2\phi/\partial t^2 = 0 \quad (18)$$

gives

$$E = \frac{\rho_0}{2}\int_V\left\{\nabla\cdot(\phi\nabla\phi)+\frac{1}{c_0^2}\left[\left(\frac{\partial\phi}{\partial t}\right)^2-\phi\frac{\partial^2\phi}{\partial t^2}\right]\right\}\mathrm{d}x\,\mathrm{d}z. \quad (19)$$

First this will be manipulated into a form to be used in the following section for calculating the energy of the vorticity. In a fluid-fixed coordinate system downstream of the airfoil the time derivatives go to zero; the divergence theorem is then used to transform to an integration over areas A_1 and A_2 giving

$$\frac{2}{\rho_0}E = \int_V \nabla\cdot(\phi\nabla\phi)\,\mathrm{d}V = -\int_{A_1}\phi w\,\mathrm{d}A+\int_{A_2}\phi w\,\mathrm{d}A, \quad (20)$$

which is essentially the form used by Ribner. However, he did not consider the detailed flow in the near vicinity of the airfoil; rather, equation (20) was used to calculate the energy in the far wake after the airfoil passed through a complete cycle, resulting in the calculation of an average energy.

To make an instantaneous energy balance between the fluid energy and the work done on the airfoil a slightly different form will be used. To this end the time derivative of equation (19) is calculated. Together with equations (16) and (18) and the fact that the volume of integration is time invariant, this gives

$$\frac{\mathrm{d}E}{\mathrm{d}t}+\int_A p\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{A} = 0, \quad (21)$$

where $\mathrm{d}\boldsymbol{A}$ represents the surface area of the volume of integration with an outward normal. This equation states that the energy change dE/dt is equal to the pressure work done on the volume. This technique was also used by Morse & Ingard (1968, p. 730) to calculate the energy radiated by moving sources. But the pressure is zero on the axis ahead of and behind the airfoil, and the velocity normal to area dA, the z-velocity normal to the airfoil, is zero on the airfoil. The product pw thus appears to be zero everywhere on areas A_1 and A_2 which are an infinitesimal distance from the axis. This dilemma is resolved by noting that the pressure becomes infinite as x approaches $-b$ from $x > -b$ and the velocity becomes infinite as x approaches $-b$ from $x < -b$, and a finite amount of energy enters the airfoil at the point $x = -b$. This is consistent with the fact that the thrust force is determined solely by the vorticity or airfoil loading in the limit as one approaches $x = -b$. If the thrust is dependent solely on the leading-edge loading, so is the energy absorbed by the airfoil and the energy subtracted from the fluid.

3.2. *Energy balance for a general disturbance*

The area of integration to be used in the application of equation (21) is shown as A_1 and A_2 in figure 2. The calculation of this area integral is simplified by noting that either p or w is zero except near the leading edge, and in this region p and w can be calculated using quasi-steady flow. Because there is an inverse square-root singularity at the leading edge, the fluid pressure and velocity will be calculated for a flat plate with a loading $\Delta p = p_u - p_l$ given by

$$\Delta p = \Delta p_0 (c/x)^{1/2}, \tag{22}$$

where the origin is now taken at the airfoil leading edge and Δp_0 has no dependence on x.

To calculate the pressure in the vicinity of the leading edge, the pressure produced by a point force $\boldsymbol{F}\delta(x)\,\delta(y)$ in a moving fluid is needed, where $\delta(x)$ is the Dirac delta function. Since quasi-steady behaviour is assumed near the leading edge the pressure is a solution of the equation

$$(\nabla^2 - M^2 \partial^2/\partial x^2) p = \nabla \cdot \boldsymbol{F}. \tag{23}$$

This is just the wave equation for a moving fluid but with the time dependence eliminated since flow around the leading edge can be treated as quasi-steady. The solution to this equation can be found by applying a Prandtl–Glauert transform, then defining $p = \nabla \cdot \boldsymbol{\Theta}$ and solving the resulting Laplace's equation for the vector $\boldsymbol{\Theta}$, obtaining the standard $\ln(r)$ source behaviour for $\boldsymbol{\Theta}$. For a z-component of force, $\boldsymbol{F} = \boldsymbol{k}F_z$, this leads to the result

$$p(x,z) = \frac{F_z}{2\pi\beta} \frac{\partial \ln \sigma}{\partial z} = \frac{F_z}{2\pi} \frac{\beta z}{\sigma^2}, \tag{24}$$

where $\sigma^2 \equiv x^2 + \beta^2 z^2$. This agrees with the small-ω expansion of the expression for an oscillating two-dimensional force (equation (47) below).

The velocity potential for this point force can be found from p using the relation for quasi-steady flow, $p = -\rho_0 U u = -\rho_0 U \partial\phi/\partial x$. Taking the derivative gives $\partial p/\partial z = -\rho_0 U \partial w/\partial x$, and integration with respect to x gives

$$w(x,z) = \frac{\partial \phi}{\partial z} = \frac{F_z}{2\pi\rho_0 U} \frac{\beta x}{\sigma^2}. \tag{25}$$

For incompressible flow this is easily verified to be the velocity produced by a bound vortex γ that produces a force on the fluid $F_z = \rho_0 U \gamma$.

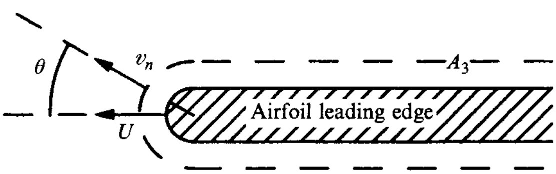

FIGURE 3. Control volume around the nose of the airfoil for calculating the work done by the fluid on the airfoil.

Integrating over the airfoil chord using equation (22) for the distribution of force near the leading edge, combined with equation (24) for the pressure produced by a point force, gives for the pressure near the leading edge

$$\frac{p(x, z\to 0)}{\Delta p_0} = \frac{\beta z c^{1/2}}{2\pi}\int_0^\infty \frac{1}{(x-\xi)^2+\beta^2 z^2}\frac{d\xi}{\xi^{1/2}} = \frac{\beta \bar{z}}{2\bar{\sigma}(\bar{\sigma}-\bar{x})^{1/2}}. \tag{26}$$

The upper integration limit is taken as ∞ rather than c since z is assumed vanishingly small and the major contribution to the integral comes from the vicinity of the lower limit. The small-z assumption is necessary because of the assumption of quasi-steady flow in equation (26). The second equality was found using integral 2.161.1 of Gradshteyn & Ryzhik (1965). The overbars on $\bar{x}$ and $\bar{\sigma}$ represent normalization by the half-chord b.

For the same force distribution and using equation (25) for the velocity produced by a point force, the z velocity component near the leading edge is

$$\frac{\rho_0 U}{\Delta p_0} w(x, z\to 0) = \frac{\beta c^{1/2}}{2\pi}\int_0^\infty \frac{x-\xi}{(x-\xi)^2+\beta^2 z^2}\frac{d\xi}{\xi^{1/2}} = -\frac{\beta(\bar{\sigma}-\bar{x})^{1/2}}{2\bar{\sigma}}, \tag{27}$$

where integral 3.252.9 of Gradshteyn & Ryzhik (1965) with $n = 0$ was used.

The product pw in equation (21) can now be calculated from equations (26) and (27), giving a surprisingly simple result. This must be integrated over both areas A_1 and A_2. The A_1 integral is

$$\frac{\rho_0 U}{(\Delta p_0)^2 b}\left(\frac{dE}{dt}\right)_{A_1} = -\frac{z\beta^2}{4}\int_{-\infty}^{\infty}\frac{dx}{\sigma^2} = -\frac{\pi\beta}{4}. \tag{28}$$

Although approximations have been made limiting the analysis to the vicinity of the leading edge, this integral has limits $-\infty < x < \infty$, since just as for equations (26) and (27), the integral is considered in the limit $z\to 0$, in which case the integrand becomes very sharply peaked around $x = 0$. Adding the equal contribution of the second area A_2, the rate of energy addition to the fluid is found to be

$$\left(\frac{dE}{dt}\right)_{fluid} = -\frac{\pi\beta b}{2}\frac{(\Delta p_0)^2}{\rho_0 U}, \tag{29}$$

where Δp_0 will generally be a function of time. On the other hand, from equations (5), (6) and (22) the rate of work done on the airfoil is found to be the negative of this, i.e. $(dE/dt)_{airfoil} = -(dE/dt)_{fluid}$. Thus, the energy taken from the fluid at any instant of time equals the energy added to the airfoil.

This is not surprising, and there is a much more direct way to look at the problem. Consider an airfoil with a small but finite leading-edge radius, as in figure 3. A control surface, A_3, fixed in the fluid, is drawn an infinitesimal distance from the airfoil surface across which the flow of fluid energy will be considered. The thrust force is the integral around the leading edge of the perturbation pressure multiplied by $\cos\theta$, to resolve the component of the normal pressure force in the axial direction; the work done on the airfoil is the thrust times the distance travelled, l. On the other hand, the rate of energy loss from the fluid is the integral of the pressure multiplied by the fluid velocity normal to the control surface. The normal fluid velocity is $U\cos\theta$, and the integral must be multiplied by l/U to find the energy added to the fluid for a distance l travelled. Thus, the integral of $p\cos\theta$ over the nose gives both the work done on the airfoil and the energy taken from the fluid.

4. Conservation of thrust work, acoustic energy and vortex energy – low frequency

Whereas the above analysis only requires the specification of the leading-edge loading and is valid for a general time behaviour, the analysis in this section assumes a sinusoidal incident gust. It was possible above to show equality between the energy absorbed by the airfoil and the energy taken from the fluid since the entire action takes place at the leading edge, and any acoustic energy is considered as part of the fluid energy. Once the leading-edge loading is specified, it is possible to calculate the thrust on the airfoil, which allows one to calculate the energy taken from the fluid. For incompressible flow the work done on the airfoil must equal the energy taken from the incident disturbance, and equation (29) is sufficient to calculate this energy exchange. For compressible flow with an incident gust disturbance, however, the energy taken from an incident vorticity is split into that remaining in the wake vorticity and the acoustic energy propagated away, but these are related to the loading distribution over the entire airfoil. To calculate this split, it thus becomes necessary to specify the incident disturbance. It will be found that the ratio of acoustic energy to energy remaining in the wake vorticity increases with an increase in the reduced frequency.

The analysis will proceed in a manner similar to that above. For calculating the acoustic energy the control surfaces A_1 and A_2 will be moved far from the x-axis. The energy of the vorticity fields upstream and downstream of the airfoil will be calculated using equation (20); this calculation will assume an infinite vortex sheet, ignoring the flow details around the airfoil. This means that only the average energy over a cycle can be determined. Since no exact solution is available for the compressible airfoil–gust interaction problem, an expansion in reduced frequency is made.

4.1. *Wake energy*

In fluid-fixed coordinates the incident gust is produced by a vortex sheet on the x-axis

$$\gamma(x) = \gamma_0 e^{ik\bar{x}}, \quad -\infty < x < \infty, \tag{30}$$

where the overbar on a coordinate, $\bar{x}$, $\bar{z}$, $\bar{\sigma}$, denotes normalization by the half-chord b. Since linearized flow is assumed, the effects of compressibility can be ignored as they are second order in the velocity. Thus, the incident vorticity gives a vertical velocity field

$$w(x) = w_0 e^{ik\bar{x}}, \quad -\infty < x < \infty, \tag{31}$$

where $w_0 = \frac{1}{2}i\gamma_0$. This has a velocity potential

$$\phi(x,z) = -bw_0 k^{-1} e^{-k\bar{z}-ik\bar{x}}, \quad -\infty < x < \infty, \quad z > 0. \tag{32}$$

Equation (20) is now used to find the average energy density of the incident vorticity. Since the integrals over A_1 and A_2 are equal, only one need be evaluated, eliminating the factor of $\frac{1}{2}$. The real parts of ϕ and w are needed in the equation. These can be expressed as

$$\frac{4}{\rho_0}E = -\int_{A_1}(\phi+\phi^*)(w+w^*)\,\mathrm{d}x = -2\int_{A_1}\phi w^*\,\mathrm{d}x, \tag{33}$$

where the asterisk represents the complex conjugate. The second equality is obtained by noting that ϕw and $\phi^* w^*$ average to zero, and the two remaining products are equal so that only one need be evaluated. Introducing equations (31) and (32), the average energy $\bar{E}_0$ for a length l of the incident vorticity is equal to that given by von Kármán & Burgers (1935, p. 307) for incompressible flow

$$\bar{E}_0 = \tfrac{1}{2}\rho_0 w_0^2 k^{-1} bl. \tag{34}$$

The incident velocity field produces a pressure on the upper airfoil surface in airfoil-fixed coordinates (Amiet 1974)

$$p_u(x,t) = -\frac{\rho_0 U w_0}{\beta}\left(\frac{b-x}{b+x}\right)^{1/2} S(k^*)\,\mathrm{e}^{\mathrm{i}[\omega t+\mu M\bar{x}+k^* f(M)]}, \tag{35}$$

where $\mu \equiv Mk^*$. Equation (35) is an approximation for $Mk^* \ll 1$. The Sears function, $S(k)$, is defined in equation (9) and $f(M)$, a function of Mach number, drops out of this analysis. The potential on the airfoil can be determined by introducing this pressure into equation (11) giving

$$\phi_u(x,t) = \frac{bw_0}{\beta} S(k^*)\,\mathrm{e}^{\mathrm{i}[\omega t-k\bar{x}+k^* f(M)]}\int_{-1}^{\bar{x}}\left(\frac{1-\xi}{1+\xi}\right)^{1/2}\mathrm{e}^{\mathrm{i}k^*\xi}\,\mathrm{d}\xi. \tag{36}$$

The axial velocity is then

$$u_u(x,t) = \frac{\partial\phi}{\partial x} = -\frac{w_0}{\beta} S(k^*)\,\mathrm{e}^{\mathrm{i}[\omega t-k\bar{x}+k^* f(M)]}\left[\mathrm{i}k\int_{-1}^{\bar{x}}\left(\frac{1-\xi}{1+\xi}\right)^{1/2}\mathrm{e}^{\mathrm{i}k^*\xi}\,\mathrm{d}\xi - \left(\frac{b-x}{b+x}\right)^{1/2}\mathrm{e}^{\mathrm{i}k^*\bar{x}}\right]. \tag{37}$$

At the trailing edge $x = b$ and the second term in the brackets becomes zero. The limits on the integral become ± 1, and this integral can be calculated in closed form. Using integrals 3.753.2 and 3.753.5 of Gradshteyn & Ryzhik (1965), the vorticity $\gamma = -2u_u$ at the trailing edge is

$$\gamma|_{x=b} = 2\pi\mathrm{i}k\beta^{-1}w_0 S(k^*)\,\mathrm{e}^{\mathrm{i}[\omega t-k+k^* f(M)]}[J_0(k^*)-\mathrm{i}J_1(k^*)]. \tag{38}$$

This vorticity is shed into the fluid, leaving in the wake a shed vorticity, expressed in fluid-fixed coordinates:

$$\gamma(x) = 2\pi\mathrm{i}k\beta^{-1}w_0 S(k^*)\,\mathrm{e}^{-\mathrm{i}k\bar{x}+\mathrm{i}k^* f(M)}[J_0(k^*)-\mathrm{i}J_1(k^*)]. \tag{39}$$

Equations (30)–(32) give the relation between vorticity, vertical velocity and velocity potential for a vortex sheet, allowing the velocity and potential produced by shed vorticity to be found from equation (39). Adding the velocity produced by the incident gust gives the overall result

$$w(x) = -w_0\{\pi k\beta^{-1}S(k^*)\,\mathrm{e}^{\mathrm{i}k^* f(M)}[J_0(k^*)-\mathrm{i}J_1(k^*)]-1\}\,\mathrm{e}^{-\mathrm{i}k\bar{x}}. \tag{40}$$

This will be introduced into equation (33), along with $\phi_u(x) = -bw(x)/k$, to find the energy in the wake. Neglecting higher-order terms

$$\phi_u(x)\,w^*(x) = -k^{-1}[\pi\beta k^*(1-\tfrac{1}{2}\pi k^*)-1]^2 bw_0^2. \tag{41}$$

Use was made of the small-k expansions of the Bessel functions and the expansion

$$S(k) = 1-\tfrac{1}{2}\pi k+\mathrm{i}k\ln(\tfrac{1}{2}k)+\mathrm{i}k\gamma+O(k^2). \quad (42)$$

Subtracting the initial wake energy to find the change in energy and simplifying,

$$\phi_u(x)\,w^*(x)-[\phi_u(x)\,w^*(x)]_0 = \pi\beta^{-1}[2-(1+\beta)\,\pi k^*]\,bw_0^2. \quad (43)$$

Introducing this into equation (33) and comparing to equation (34), the change in average energy for a length l of the wake compared to the energy of the incident vorticity is

$$\Delta\bar{E}_{vorticity}/\bar{E}_0 = -\pi\beta(2-\pi k^*-\beta\pi k^*)\,k^*, \quad (44)$$

where $k^* \equiv k/\beta^2$. From equation (14), the work done on the airfoil on traversing a length l is

$$\bar{E}_{thrust}/\bar{E}_0 = \bar{T}l/\bar{E}_0 = 2\pi\beta(1-\pi k^*)\,k^*. \quad (45)$$

Comparing the two equations, one can deduce that there is an average energy deficit of

$$\bar{E}_{acoustic}/\bar{E}_0 = (1-\beta)\,\beta\pi^2k^{*2}. \quad (46)$$

This must go into acoustic propagation as will be shown below.

Finally, it should be emphasized that equations (44)–(46) are not valid for $M\rightarrow 1$ because of the limitation $Mk^* \ll 1$. In practice, the airfoil response approximation remains reasonably accurate for $Mk^* \leqslant \pi/4$ (Amiet 1974, 1976). The case of $M\rightarrow 1$ is considered in the high-frequency case below in §5.

4.2. *Acoustic energy*

To calculate the radiated acoustic energy to the same order as equation (46), only the lowest-order lift solution is needed and the airfoil sound source can be treated as compact; all trailing-edge effects are included in the left response. The far-field pressure and velocity will be determined in order to calculate the acoustic energy radiated using equation (21), where the surfaces A_1 and A_2 are now assumed to be located in the far field.

The general expression for the pressure due to a point force $\boldsymbol{F}\delta(x)\,\delta(z)\exp(\mathrm{i}\omega t)$, in an airfoil-fixed coordinate system, is (see e.g. Fung 1969, equations (14.1.18) and (14.2.12))

$$p(x,z) = \frac{\mathrm{i}\boldsymbol{F}}{4\beta}\cdot\nabla[H_0^{(2)}(\mu\bar{\sigma})\,\mathrm{e}^{\mathrm{i}\mu M\bar{x}}]\,\mathrm{e}^{\mathrm{i}\omega t}. \quad (47)$$

Assuming a vertical force $\boldsymbol{F} = \boldsymbol{j}F_y$ and expanding for large σ using the large-argument expansion of the Bessel function the expression for far-field pressure is

$$p(x,z) = \frac{F_z}{4}\frac{z}{\sigma}\left(\frac{2kM}{\pi\sigma b}\right)^{1/2}\mathrm{e}^{\mathrm{i}\mu(M\bar{x}-\bar{\sigma})+\mathrm{i}(\omega t+\pi/4)}. \quad (48)$$

The velocity is related to the pressure in the far field by the plane wave relation $p = \rho_0 c_0|v|$. In equation (21) the vertical component of velocity is needed. Since the magnitude of the velocity is known, all that is needed is the direction of the velocity vector. This vector is normal to the wavefront, and so can be found by taking the gradient of the phase in equation (48). The phase ψ, with a proportionality constant C, is

$$\psi = C(Mx-\sigma). \quad (49)$$

Taking the gradient gives for the wavefront normal

$$\boldsymbol{n} = [(M\sigma - x)\boldsymbol{i} - z\beta^2\boldsymbol{k}]/(\sigma - Mx). \tag{50}$$

Thus, the pressure must be multiplied by $z\beta^2/[(\sigma - Mx)\rho_0 c_0]$ to find the z-component of the vertical velocity and

$$\left(\frac{8\rho_0 c_0^2 \pi}{\beta^2 F_z^2\, \omega}\right) pw^* = \frac{z^3}{\sigma^3}\frac{1}{\sigma - Mx}. \tag{51}$$

This is introduced into equation (21), requiring an x-integration. The following integral was found using the symbolic algebra computer program Maple (Char *et al.* 1991), and was verified numerically:

$$\int_{-\infty}^{\infty} [(x^2+\beta^2)^{3/2}((x^2+\beta^2)^{1/2} - Mx)]^{-1}\,\mathrm{d}x = \frac{\pi(1-\beta)}{M^2\beta^3}. \tag{52}$$

Equation (51) was derived assuming a point force for F_z, but since the source can be assumed to be compact to the order calculated here, F_z can be taken to be equal to the airfoil lift given by equation (1) with $\alpha = w_0/U$. Combining this with equations (21) and (51)–(52), multiplying by a factor of 2 to account for integration over both the surfaces A_1 and A_2 and a factor of $\frac{1}{2}$ since $\frac{1}{4}(p+p^*)(w+w^*)$ averages to $\frac{1}{2}pw^*$, gives

$$(\mathrm{d}\bar{E}/\mathrm{d}t)_{acoustic} = (1-\beta)\beta\pi^2 k^{*2} U\bar{E}_0/l, \tag{53}$$

where the average, $\bar{E}$, denotes the energy of the fluid averaged over one cycle. Finally, the time for the airfoil to travel a distance l is l/U, and the acoustic energy passing through the integration surface during this time is again found to be given by equation (46) which was the difference between the energy subtracted from the vorticity and the work done on the airfoil.

Thus, for small k and relative to the energy in the incident vorticity, $\bar{E}_0$, equation (44) shows that the energy subtracted from vorticity is $O(k)$, as is the work done on the airfoil given by equation (45), and equation (46) shows that the energy in the acoustic radiation is $O(k^2)$.

5. Conservation of thrust work, acoustic energy and vortex energy – high frequency

5.1. *Wake energy*

The pressure on an airfoil at high frequency ($Mk^* \gg 1$) is (Amiet 1976)

$$\frac{p_u(x)}{\rho_0 U w_0} = -\frac{1}{[\pi k\bar{x}(1+M)]^{1/2}}\,\mathrm{e}^{-\mathrm{i}\mu(1-M)\bar{x} - \mathrm{i}\pi/4}, \tag{54}$$

where the airfoil-fixed coordinate system has $x = 0$ at the leading edge. This pressure is derived assuming the airfoil is a semi-infinite flat plate, thus ignoring the trailing-edge effects. A trailing-edge correction is also given in the reference, but this is $O(k^{-1})$, and so is small compared to the terms retained in this analysis. Landahl (1961) shows that iterating between the leading and trailing edges in this manner produces a proper convergent series. The potential given by introducing equation (54) into equation (11) is

$$\frac{\phi_u(x)}{bw_0} = \frac{\mathrm{e}^{-\mathrm{i}(k\bar{x}+\pi/4)}}{[\pi k(1+M)]^{1/2}}\int_0^{\bar{x}} \mathrm{e}^{\mathrm{i}k\xi(1-M)/\beta^2}\frac{\mathrm{d}\xi}{\xi^{1/2}} = \frac{\mathrm{e}^{-\mathrm{i}k\bar{x}}}{k} + O(k^{-3/2}). \tag{55}$$

The approximate result was obtained by setting the upper limit to infinity; the next term in an expansion can readily be found by writing the integral as a sum of an integral with limits of $\pm\infty$ minus an integral with limits of x and ∞, but this is not needed. Using equations (31) and (32) to relate ϕ and w shows that this potential field produces a velocity $w = -w_0 \exp(-ikx)$; that is, it is just the negative of the incident field and there is complete cancellation of the incident vorticity. All the energy in the incident vorticity goes into thrust work on the airfoil and acoustic propagation. The complete cancellation of the incident vorticity was also noted by Amiet (1990) for the incompressible high-frequency case; see equation (52) and surrounding text where some discussion of the physical reasons for this cancellation is given. The complete cancellation occurs only because the incident vorticity lies on the axis, the same as the airfoil. If the incident vorticity were to lie off the axis, far downstream one would have the incident vorticity, off the axis, together with shed vorticity on the axis that gives a flow field equivalent to the mirror image of the incident vorticity.

5.2. *Acoustic energy*

Equation (48) gives the acoustic radiation for a point force at the origin (0, 0) with the observer at (x, z). Since compactness is no longer assumed, when this equation is used to represent a force on the airfoil, it must be modified to give the far-field pressure produced by a force at $(x_0, 0)$. When integrating over the source strength, x_0 becomes the integration variable in equation (57) below. Substituting $x - x_0$ for x in equation (48) and expanding for $x \gg x_0$ gives

$$p(x,z) = \frac{F_z}{4}\frac{z}{\sigma}\left(\frac{2\omega}{\pi\sigma c_0}\right)^{1/2} \mathrm{e}^{\mathrm{i}\mu[M(\bar{x}-\bar{x}_0)-\bar{\sigma}+\bar{x}_0 x/\sigma]+\mathrm{i}(\omega t+\pi/4)}. \tag{56}$$

Using $F_z = \Delta p = 2p_u$ with p_u given by equation (54) and integrating over the airfoil gives for the far-field sound

$$\frac{p(x,z)}{\rho_0 U w_0} = -\frac{1}{2\pi}\frac{z}{\sigma}\mathrm{e}^{\mathrm{i}\mu(M\bar{x}-\bar{\sigma})+\mathrm{i}\omega t}\left[\frac{2Mb}{\sigma(1+M)}\right]^{1/2}\int_0^2 \mathrm{e}^{-\mathrm{i}\mu\xi(1-x/\sigma)}\frac{\mathrm{d}\xi}{\xi^{1/2}}. \tag{57}$$

For the lowest-order term in a k^{-1} expansion, the upper limit of the integral can be replaced by ∞ giving

$$\frac{p(x,z)}{\rho_0 U w_0} \sim -\frac{z}{\sigma}\left[\frac{b(1-M)}{2\pi k(\sigma-x)}\right]^{1/2} \mathrm{e}^{\mathrm{i}\mu(M\bar{x}-\bar{\sigma})+\mathrm{i}(\omega t-\pi/4)}. \tag{58}$$

Again using equation (50) for the normal to the wavefront, the z-component of far-field velocity is

$$\frac{w(x,z)}{w_0} \sim -\left[\frac{b(1-M)}{2\pi k(\sigma-x)}\right]^{1/2}\frac{z^2 M\beta^2}{\sigma(\sigma-Mx)}\mathrm{e}^{\mathrm{i}\mu(M\bar{x}-\bar{\sigma})+\mathrm{i}(\omega t-\pi/4)}. \tag{59}$$

The product pw^* is then

$$\frac{pw^*(x,z)}{\rho_0 U w_0^2} \sim \frac{bz^3 M\beta^2(1-M)}{2\pi k\sigma^2(\sigma-x)(\sigma-Mx)}. \tag{60}$$

The following integral was found using the symbolic algebra computer program, Maple (Char *et al.* 1991), and verified numerically

$$\int_{-\infty}^{\infty} [(x^2+\beta^2)^{1/2}-x)((x^2+\beta^2)^{1/2}-Mx)(x^2+\beta^2)]^{-1}\,\mathrm{d}x = \pi\frac{1+M-\beta}{M\beta^4}. \tag{61}$$

Performing the x-integration in equation (21), multiplying the result by 2 to account for both surfaces A_1 and A_2 and by $\frac{1}{2}$ since $\frac{1}{4}(p+p^*)(w+w^*)$ averages to $\frac{1}{2}pw^*$, gives

$$\frac{\mathrm{d}\bar{E}_{acoustic}}{\mathrm{d}t} = Ul^{-1}\left(1-\left(\frac{1-M}{1+M}\right)^{1/2}\right)\bar{E}_0. \tag{62}$$

The energy transmitted during a time $\Delta t = l/U$ is then

$$\bar{E}_{acoustic}/\bar{E}_0 = 1-(1-M)/\beta. \tag{63}$$

The thrust is given by equation (6) with G given by equation (5). The value for G using the particular loading for the sinusoidal gust considered here is found by using equation (54) for Δp. This gives

$$|G| = 4(w_0/U)[2\pi k(1+M)]^{-1/2}. \tag{64}$$

Using this in equation (6) gives for the average work done on the airfoil on moving a distance l

$$\bar{E}_{thrust}/\bar{E}_0 = \bar{T}l/\bar{E}_0 = (1-M)/\beta. \tag{65}$$

The average energy of the incident gust is given by equation (34) and one notes that the energy in the incident gust is equal to the acoustic energy radiated plus the energy taken out by the work on the airfoil. In contrast to the low-frequency case, however, now all the incident vortex energy is transformed into acoustic energy and work on the airfoil, and these three energies are of the same order in k. There is still some energy in the wake, but this is of higher order in k^{-1}. One also notes from equations (63) and (65) that as $M \to 1$ the energy of the incident vorticity is transformed more and more into acoustic radiation until at $M = 1$ there is complete transformation of the energy of the incident vorticity into acoustic radiation. Guo (1989) also finds the energy radiated to increase monotonically with increasing M for the intersection of a semi-infinite airfoil with a circular jet, but in that case the function of Mach number is somewhat more complex than equation (63).

This behaviour can be explained physically. The airfoil response given by equation (54) is for a semi-infinite airfoil with no trailing edge. This explains why there is no vorticity left in the wake, to this order. As the gust impinges on the airfoil and moves far downstream from the leading edge, the flow field becomes that of the incident gust plus its mirror image in the airfoil, which, because the airfoil now appears infinite, exactly cancels an incident gust lying on the axis. The acoustic wave created at the leading edge impinges on the trailing edge, scattering the wave and creating shed vorticity. This is demonstrated by a trailing-edge correction, but the result is $O(k^{-1/2})$ smaller than the acoustic wave produced by the leading edge (Landahl 1961). For $M < 1$, the incident gust as it impinges on the airfoil does not immediately see an infinite flat plate, so the mirror-image solution is not instantaneously created. Rather, the gust has time to interact with the leading edge, creating the suction force. However, at $M = 1$, the gust no longer has the time to interact with the leading edge. A pressure perturbation created on the upper surface of the airfoil can no longer create a flow around the leading edge to relieve the pressure; the airfoil instantly appears as an infinite flat plate to that portion of the gust cut by the airfoil. This is repeated at the trailing edge where the acoustic wave created at the leading edge impinges on the trailing edge, but no radiating dipoles can appear just upstream of the trailing edge since the presence of the trailing edge cannot be propagated upstream.

Other recent papers considering energy conservation for similar problems are Ffowcs Williams & Guo (1988), Guo (1989, 1991*a*, *b*) and Levine (1991). Since three of the references consider supersonic flow, there is no infinity in the velocity at the leading edge and so no leading-edge suction force. The results of Guo (1989) are for subsonic flow, and it would be interesting to compare them with the present results. However, Guo calculates the energy balance for the case of a circular jet impinging on an airfoil while the present paper considers the two-dimensional problem of the energy radiated by an incident sinusoidal gust, and there is no obvious quantitative comparison between the results for the two cases.

6. Conclusions

The thrust force on a flat-plate airfoil encountering a gust can be calculated using the instantaneous leading-edge flow. The pressure and velocity of the fluid in the vicinity of the leading edge depend only on the asymptotic limit of the airfoil loading at the leading edge. Integrating the product of this pressure and velocity over a control surface parallel to the x-axis and an infinitesimal distance above and below the airfoil shows that the energy subtracted from the fluid is equal to the work done on the airfoil. The equality between the work done on the airfoil and the energy subtracted from the fluid becomes an almost trivial statement, however, for a control volume an infinitesimal distance from an airfoil with a small rounded leading edge. For incompressible flow the energy difference between incident and wake vorticity equals the work done on the airfoil, but for compressible flow this energy difference equals the sum of the work done on the airfoil and acoustic energy radiated away, as Ribner noted that it must. When the Mach number goes to one, the work done on the airfoil also goes to zero, and all energy taken from the incident vorticity is radiated as acoustic energy. When measured relative to the energy of the incident vorticity, at low frequency the energy difference between incident and wake vorticity is $O(k)$ as is the energy added to the airfoil while the radiated acoustic energy is $O(k^2)$. At high frequency, the energy of the incident vorticity, the energy added to the airfoil and the acoustic radiation are all of the same order, and to this order no energy remains in the wake, any remaining wake energy being $O(k^{-1/2})$ smaller. This cancellation of the incident vorticity only occurs if the incident vorticity lies in the plane of the airfoil. If the incident vorticity does not lie in the plane of the airfoil, far downstream the flow field will consist of the incident vorticity plus its mirror image, which gives only partial cancellation of the incident flow energy.

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